Parameterized Complexity of Generalized Vertex Cover Problems

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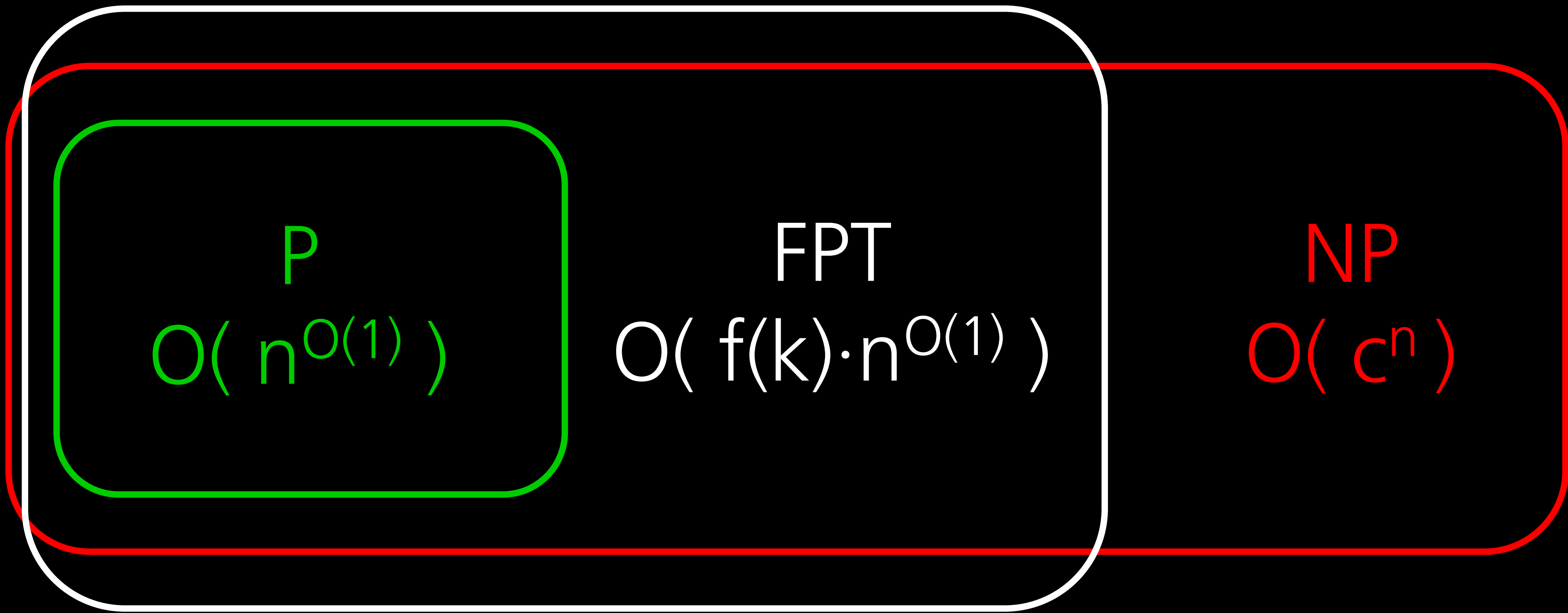
Parameterized Complexity

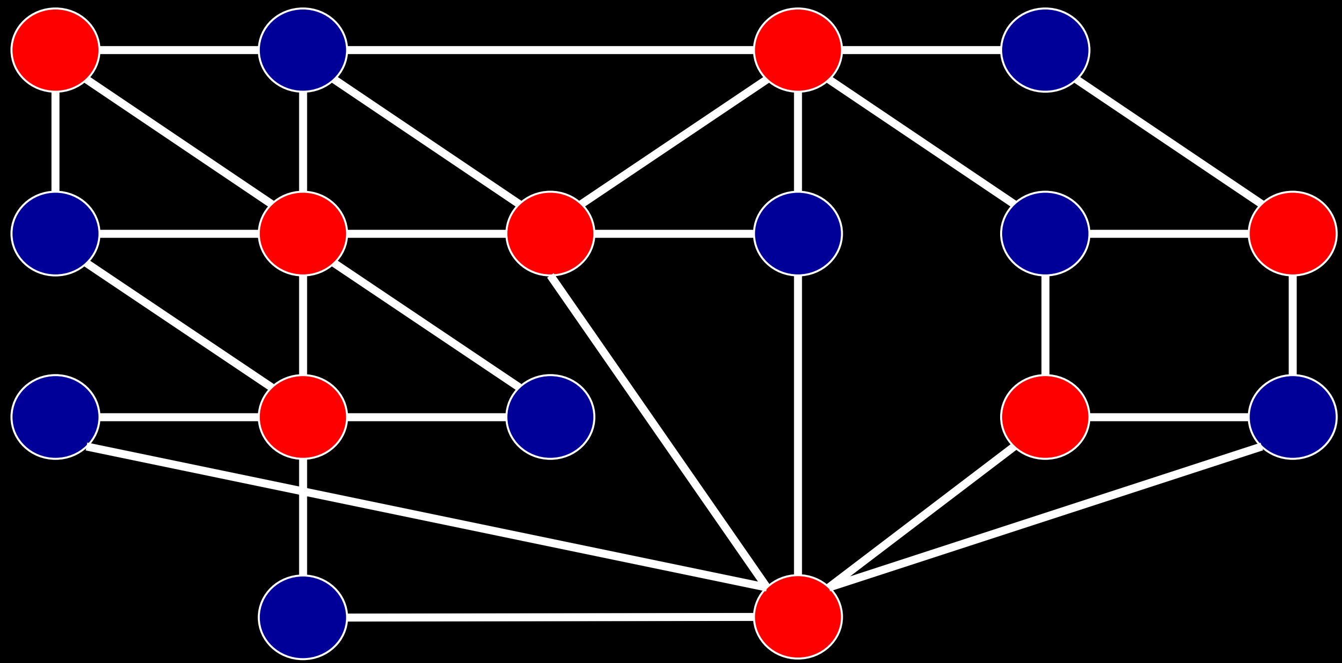


Two-dimensional approach:
Problem-size *n*, parameter *k*.

Parameterized Complexity

Intractability	Basic class of fixed-parameter intractability: W[1]
Kernels	If a problem is fixed-parameter tractable, we can reduce an instance of size n to an instance of size f(k).

VERTEX COVER (VC)



Given a graph G, find a set C of at most k vertices such that each edge has at least one neighbor in C.

Generalizations of VC

Connected VC	Set C must be connected in G
Tree Cover	Set C must induce a cheap tree
Tour Cover	Set C must induce a cheap tour
Capacitated VC	Vertices have covering capacity
Soft Capacitated VC	Covering capacity can be "bought"
Hard Capacitated VC	Covering capacity is fixed
Max Partial VC	Cover as many edges as possible
Min Partial VC	Cover as few edges as possible

Well-Studied Approximability

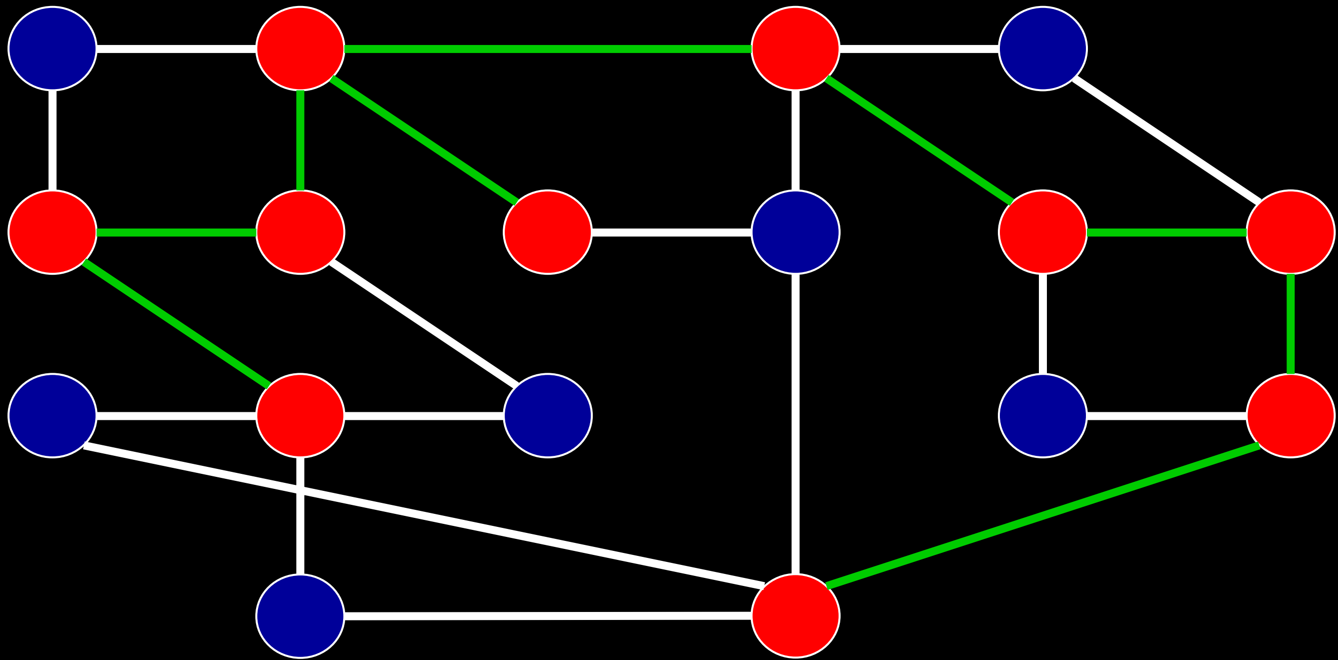
CONNECTED VC	Factor 2 [Arkin et al., *Inform. Process. Lett.*, 1993]
TREE COVER	Factor 3 [Könemann et al., *Algorithmica*, 2004]
TOUR COVER	Factor 3 [Könemann et al., *Algorithmica*, 2004]
CAPACITATED VC	Factor 2 [Guha et al., *J. Algorithms*, *2003*]
SOFT CAPACITATED VC	?
HARD CAPACITATED VC	Factor 2 [Gandhi et al., *Proc. 30th ICALP*, 2003] weighted: at least as hard as set cover [Chuzhoy, *Proc. 43rd FOCS,* 2002]
MAX PARTIAL VC	Factor 2 [Bshouty et al., *Proc. 15th STACS*, 1998]
MIN PARTIAL VC	?

New FPT Results

CONNECTED VC	$6^k n + 4^k n^2 + 2^k n^2 \log(n) + 2^k nm$
TREE COVER	$(2k)^k\ km$
TOUR COVER	$(4k)^k\ km$
CAPACITATED VC	$1.2^{k^2} + n^2$
SOFT CAPACITATED VC	$1.2^{k^2} + n^2$
HARD CAPACITATED VC	$1.2^{k^2} + n^2$
MAX PARTIAL VC	W[1] – hard
MIN PARTIAL VC	W[1] – hard

Focus: Differentiation between
fixed-parameter tractabiliy and W[1]-hardness

CONNECTED VC and Variants



CONNECTED VERTEX COVER: Given a graph G=(V,E) and an integer $k \geq 0$, determine whether there exists a vertex cover C for G containing at most k vertices such that the subgraph of G induced by C is connected.

CONNECTED VC and Variants

Algorithm (Connected VC)

1. Enumerate minimal Vertex Covers C (with at most k vertices)
2. If a C is connected, terminate
3. Otherwise, compute a Steiner Minimum Tree which has the vertices in C as its endpoints (YES-instance if this tree has at most k-1 edges)

Runtime

$$\underbrace{2^k}_{1.\,+\,2.} \cdot \underbrace{(\, 3^k n + 2^k n^2 + n^2 \log(n) + nm \,)}_{3.}$$

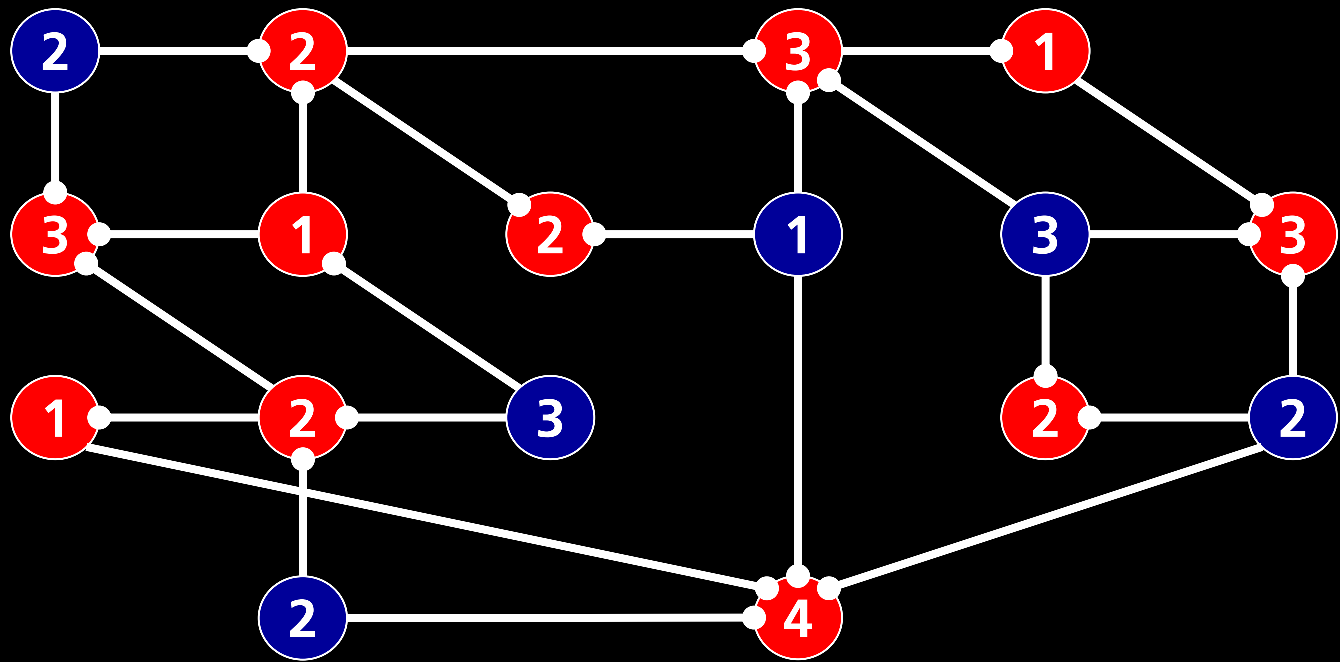
CONNECTED VC and Variants

Algorithm

1. Enumerate minimal Vertex Covers C (with at most k vertices)
2. If a C is connected, terminate
3. Otherwise, compute a Steiner Minimum Tree which has the vertices in C as its endpoints (YES-instance if this tree has at most k-1 edges)

Different steps 2 and 3 here yield $(2k)^k$ km algorithm for TREE COVER and $(4k)^k$ km algorithm for TOUR COVER

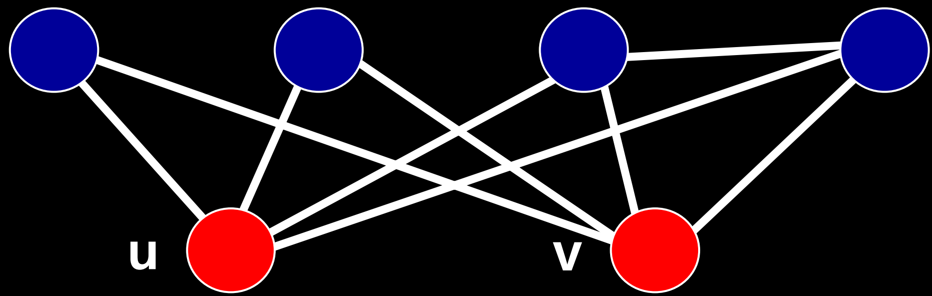
CAPACITATED VC



CAPACITATED VERTEX COVER: Given a vertex-weighted (with positive real numbers) and capacitated graph G=(V,E), an integer $k \geq 0$, and a real number $W \geq 0$, determine whether there exists a capacitated vertex cover C for G containing at most k vertices such that $\Sigma_{v \in C}\, w(v) \leq W$.

Capacitated VC

Reduction to a Problem Kernel (Uniform Vertex Weights)



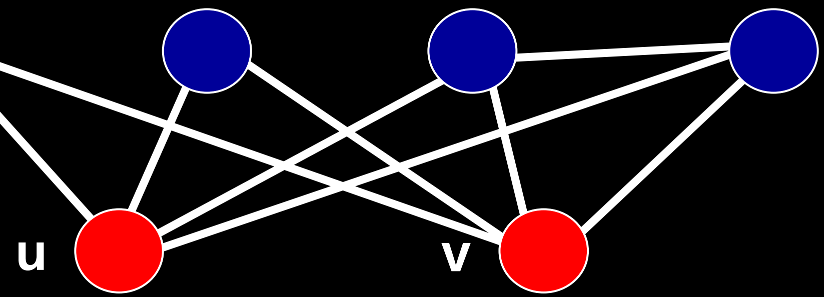
Observation

Let u and v be two unconnected vertices with the same open neighborhood.

If v has more capacity than u, then u is in a minimum capacitated vertex cover only if v is as well.

CAPACITATED VC

Reduction to a Kernel
(Uniform Vertex Weights)



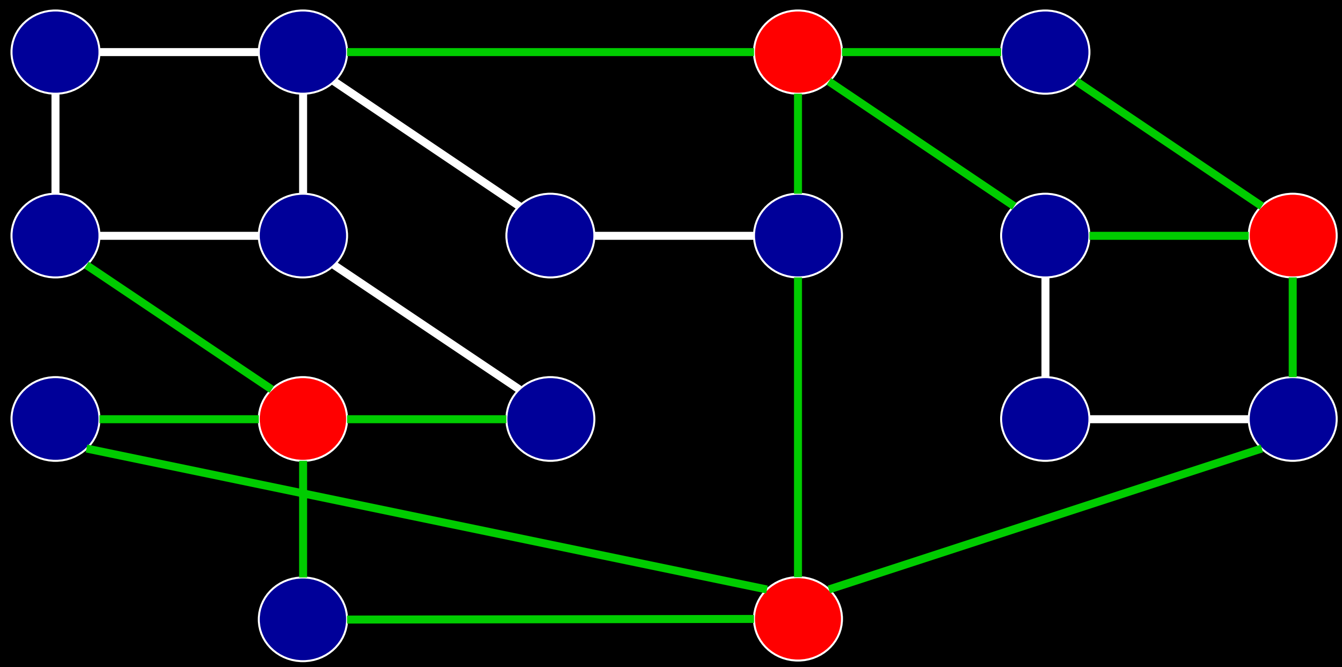
Reduction Rule (RR)	If more than k independent vertices have the same neighbor set, the one with minimum capacity may be removed.
Kernel	Start with a linear-time factor-2 approximation. Then, there are at most 2^{2k} neighbor sets for these independent vertices using the RR.

CAPACITATED VC

THEOREM: Given an n-vertex graph G=(V,E) and an integer $k \geq 0$ as an input instance of weighted CVC, an $O(4^k k^2)$ problem kernel can be constructed in $O(n^2)$ time. In the case of uniform vertex weights, the kernel has only $O(4^k k)$ vertices.

Algorithm	Problem kernel already proves that CAPACITATED VC is fixed-parameter tractable (brute force).
Speedup	Using an idea also based on the structure of neighborhoods yields an $O(1.2^{k^2} + n^2)$ algorithm.
Hard/Soft	Our ideas also hold for the HARD / SOFT variants of CAPACITATED VERTEX COVER.

MAXIMUM PARTIAL VC



MAXIMUM PARTIAL VERTEX COVER: Given a graph $G=(V,E)$ and two integers $k \geq 0$ and $t \geq 0$, determine whether there exists a vertex subset $V' \subseteq V$ of size at most k such that V' covers at least t edges.

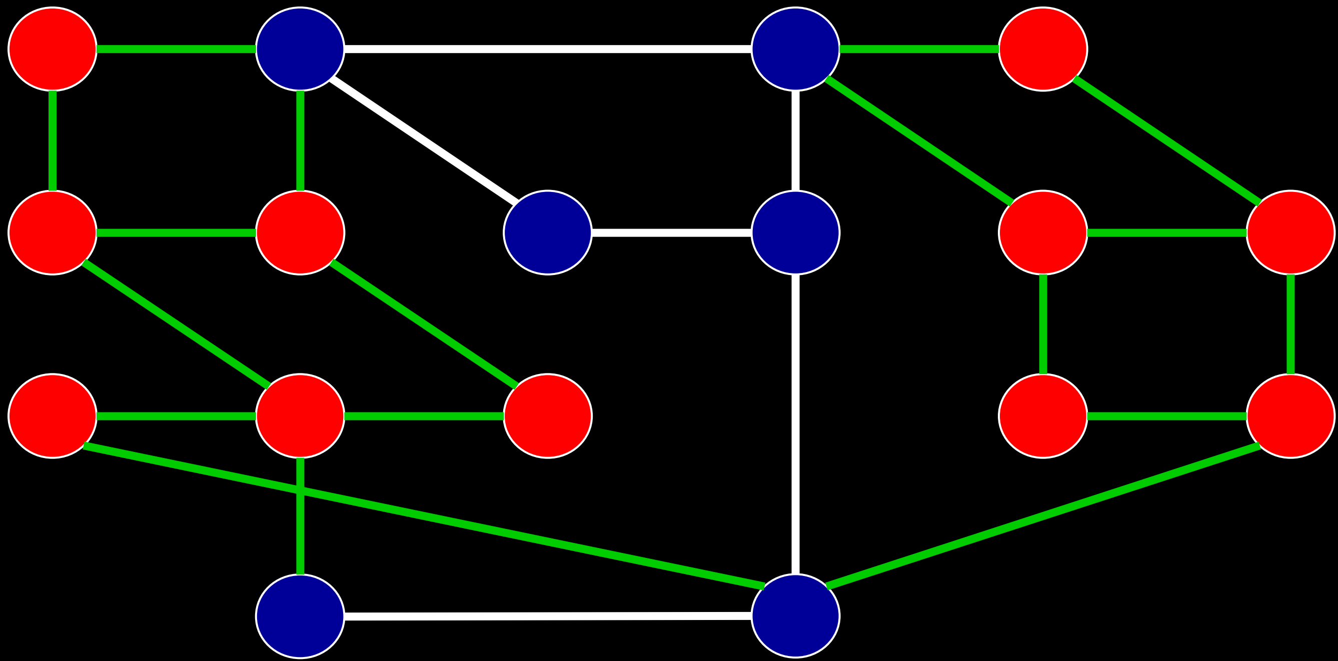
MAXIMUM PARTIAL VC

MAXIMUM PARTIAL VERTEX COVER: Given a graph G=(V,E) and two integers $k \geq 0$ and $t \geq 0$, determine whether there exists a vertex subset $V' \subseteq V$ of size at most k such that V' covers at least t edges.

Observation Relationship to the W[1]-hard INDEPENDENT SET problem: If we find high-degree independent vertices, they should be in an optimal solution.

Hardness Indeed, parameterized reduction from INDEPENDENT SET shows the W[1]-hardness of PARTIAL VC.

MINIMUM PARTIAL VC



Analogy

For MINIMUM PARTIAL VC, we want to cover as few edges as possible. Intuitively, if the graph is regular, we need to find dense regions in it to achieve this. Hence, W[1]-hardness is shown by a reduction from CLIQUE.

Conclusion

Summary — Extension and completion of the fpt-picture for natural variants and generalizations of VERTEX COVER.

To Do — Improve the running time and learn more about amenability to kernelization and data reduction techniques.

What We Do — We are seeing how the problems behave with parameters other than solution size (e.g., treewidth).