Probe Matrix Problems: Totally Balanced Matrices

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Abstract. Let $\mathcal{M}$ be a class of 0/1-matrices. A $0/1/\star$-matrix A where the $\star$s induce a submatrix is a *probe matrix* of $\mathcal{M}$ if the $\star$s in A can be replaced by 0s and 1s such that A becomes a member of $\mathcal{M}$. We show that for $\mathcal{M}$ being the class of totally balanced matrices, it can be decided in polynomial time whether A is a probe totally balanced matrix. On our route toward proving this main result, we also prove that so-called partitioned probe strongly chordal graphs and partitioned probe chordal bipartite graphs can be recognized in polynomial time.

1 Introduction

With this paper, we bring together two lines of research. On the one hand, we consider totally balanced matrices and the closely related strongly chordal graphs, and, on the other hand, we study sandwich and, more specifically, probe problems. We provide first positive results on the recognizability of probe totally balanced matrices and, correspondingly, partitioned probe chordal bipartite and partitioned probe strongly chordal graphs.

Sandwich Problems. Sandwich problems are studied in graph and hypergraph theory as well as for matrix problems [10, 13, 14, 16]. For a graph property Π, the corresponding sandwich problem is defined as follows: Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs such that $E_1 \subseteq E_2$. Is there a graph $G = (V, E)$ such that G satisfies Π and $E_1 \subseteq E \subseteq E_2$? Similarly, in case of matrices one is given a matrix with entries from $\{0, 1, \star\}$, and one asks whether one can replace the $\star$s by 0s and 1s such that the matrix fulfills a given property (such as being totally balanced).

Sandwich problems can be seen as generalizations of recognition and completion problems; for instance, graph completion problems allow the arbitrary

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addition of edges whereas in sandwich problems the addition of certain edges (those not in E_2) is disallowed. Unfortunately, as a rule, sandwich problems are notoriously hard. For instance, in their classic paper, Golumbic, Kaplan, and Shamir [14, Figure 3] have pointed out the NP-completeness of sandwich problems for many subclasses of perfect graphs. Very recently, Faria *et al.* [12] announced the NP-completeness of the sandwich problem for strongly chordal graphs. Analogous results hold for matrix sandwich problems [13, 16]. For instance, considering the class of matrices with the consecutive ones property (see for example [4]), the corresponding sandwich problem is NP-complete [16].[1]

With the *probe* concept, motivated by applications in computational biology, a new, seemingly more tractable[2] sandwich concept entered the stage. It already received considerable attention in the graph-algorithms community; for example, see [3, 5, 15]. Given a class of graphs $\mathcal{G}$, a graph G is a *probe graph of* $\mathcal{G}$ if its vertices can be partitioned into two sets $\mathbb{P}$ (the *probes*) and $\mathbb{N}$ (the *nonprobes*), where $\mathbb{N}$ is an independent set, such that G can be *embedded* into a graph of $\mathcal{G}$ by adding edges between certain nonprobe vertices. Notice that this is a special version of graph sandwich problems with $G_1 = G$ and $G_2 = (V, E \cup E')$ where E' contains the edges between all vertex pairs from $\mathbb{N}$.

0/1-Matrix Problems. Interpreting 0/1-matrices as adjacency matrices, there is a direct connection between matrix and graph problems. In what follows, we will make extensive use of this close relationship, setting out our proofs in terms of graph theory rather than matrix theory. The original motivation for our work, however, comes from matrices and integer linear programming.

It is well-known that when the matrix of a linear program is balanced, totally balanced, or totally unimodular, then the corresponding integer linear programming problem can be solved in polynomial time. The general case is NP-hard (see, *e.g.*, [4, 21]). The study of balanced 0/1-matrices, that is, matrices where no square submatrix of odd order contains exactly two 1s per row and per column, goes back to Berge [2]. In particular, he proved that a 0/1-matrix is balanced if and only if the corresponding bipartite graph has no induced cycle of length 2 mod 4. Later, Lovász suggested to study totally balanced 0/1-matrices, that is, matrices that correspond to bipartite graphs without any induced cycle of length more than 4. In this paper, we want to initiate a study of probe problems referring to these matrices. More precisely, we focus on the perhaps simplest case, that is, totally balanced matrices. This matrix class, a subclass of balanced matrices, finds applications in various contexts [1, 4]. We employ the following, for our purposes most suitable definition of totally balanced matrices (due to Lovász). To this end, note that a 0/1-matrix uniquely corresponds to a bipartite graph where one color class stands for the rows and the other for the columns.

A 0/1-matrix is *totally balanced* if its corresponding bipartite graph is *chordal bipartite*, that is, if the bipartite graph has no chordless cycle of length more than

[1] The matrices with the consecutive ones property form a subclass of totally balanced matrices.

[2] For instance, whereas the sandwich problem for chordal graphs is NP-complete [14], the corresponding probe chordal problem is polynomial-time solvable [15, 3].

four. Note that the recognition of chordal bipartite graphs has a long history, see Huang [18] for recent characterizations. The sandwich problem for totally balanced matrices is, given a matrix A with entries from $\{0, 1, \star\}$, try to replace the $\star$s with 0s and 1s such that A becomes totally balanced. Unfortunately, this problem turns out to be NP-complete [12]. Hence, instead we somewhat naturally "relax" the problem formulation, considering its probe version. Let A be a $0/1/\star$-matrix in which the $\star$s induce a *submatrix*. Then A is called *probe totally balanced* if the $\star$s in A can be replaced by 0s and 1s such that A becomes totally balanced.

Seen from a more general perspective, probe matrices stand in one-to-one correspondence with *partitioned* probe bipartite graphs; partitioned means that the partition of the vertices into probes and nonprobes is part of the input. More precisely, exactly those rows and columns that contain at least one $\star$-entry correspond to the nonprobe vertices.

The main result of this work in terms of matrices is to show that one can decide in polynomial time whether a given $0/1/\star$-matrix is probe totally balanced, and, if so, find a corresponding replacement of the $\star$s by 0s and 1s. This can also be considered as a step toward solving the corresponding recognition problems for probe balanced and probe totally unimodular matrices.

Due to the lack of space, some proofs are omitted.

2 Preliminaries

For notational convenience, for sets A and B and elements x we write $A + B$, $A - B$, $A + x$, and $A - x$ as shorthands for $A \cup B$, $A \setminus B$, $A \cup \{x\}$, and $A \setminus \{x\}$. Moreover, for a graph $G = (V, E)$ we denote by $G - x$ the induced subgraph $G[V - x]$. The *complement graph* $\bar{G} = (V, E')$ of a graph $G = (V, E)$ is given by $E' := \{\{u, v\} \mid u, v \in V,\ \{u, v\} \notin E\}$. For a vertex x we denote by $N_G(x)$ the set of its neighbors in graph G and we let $N_G[x] = N_G(x) + x$ be its closed neighborhood. For a subset $A \subseteq V$ we write $N_G(A) = \bigcup_{x \in A} N_G(x) - A$. Herein, we omit the subscript "G" when it is clear from the context. A subset I of vertices is called an *independent set* if the induced subgraph $G[I]$ has no edge, whereas a subset K of vertices is called a *clique* if $G[K]$ has all possible edges.

A *chord* in a (simple) cycle is an edge connecting two vertices of the cycle which are not adjacent in the cycle. A graph is *chordal* if it has no chordless cycle of length more than 3. A chord in an even cycle is *odd* if the distance between the endvertices along the cycle is odd.

The major technical contribution of this paper is based on the close relationship between the class of strongly chordal graphs and totally balanced matrices. The following is not the original definition of strong chordality but a characterization due to Farber [11] which best fits our purposes.

Definition 1 ([11]). *A graph is* strongly chordal *if it is chordal and every even cycle of length at least six has an odd chord.*

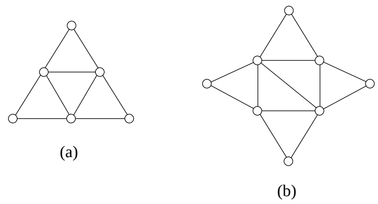

Fig. 1. A 3-sun (a) and a trampoline (b).

Polynomial-time recognition algorithms for strongly chordal graphs (based on doubly-lexical orderings) are due to Lubiw [19] and Paige and Tarjan [20]. Strongly chordal graphs have a useful characterization based on the following notion.

Definition 2. *A* simple vertex *is a vertex* x *such that for every pair* $y, z \in N(x)$ *either* $N[y] \subseteq N[z]$ *or* $N[z] \subseteq N[y]$.

That is, the closed neighborhoods of the neighbors of a simple vertex form a chain under inclusion. Notice that a simple vertex is *simplicial*, *i.e.*, its neighborhood induces a clique.

Theorem 1 ([11]). *A graph is strongly chordal if and only if every induced subgraph has a simple vertex.*

The class of strongly chordal graphs can be characterized by forbidden induced subgraphs called suns. A *sun* is a graph obtained from an even cycle of length at least 6 by adding edges to make a maximum independent set into a clique. If the cycle is of length $2k$, then the sun is called a k-*sun*. We call the set of vertices of degree 2 in a sun the *independent set of the sun*, and the set of vertices of degree at least 4, the *clique of the sun*. If some edges of the clique are missing but the graph is still chordal, it is called a *trampoline*. Figure 1 shows a 3-sun and a trampoline.

Theorem 2 ([6, 11]). *A graph is strongly chordal if and only if it is chordal and has no induced sun.*

3 Partitioned Probe Strongly Chordal Graphs

Strongly chordal graphs are closely related to totally balanced matrices. As a basis for showing in Sect. 4 that probe totally balanced matrices can be recognized in polynomial time, here we show how to recognize partitioned probe strongly chordal graphs, or PP-strongly chordal graphs for short, in polynomial time.

In what follows, an *embedding* of a probe graph G into a graph class $\mathcal{G}$ always means a graph contained in $\mathcal{G}$ which is obtained from G by adding edges between the nonprobes in G. In our recognition algorithm the first two steps concentrate on finding an embedding of the partitioned probe graph into a chordal graph. In a third step we take care of the suns (cf. Theorem 2) in this chordal embedding.

To find a chordal embedding, we first prove that we only have to destroy chordless 4-cycles. More precisely, we show that probe strongly chordal graphs are *weakly chordal*. That is, they contain neither an induced *hole* nor an induced *antihole*. A hole is a chordless cycle of length at least 5. An antihole is the complement of such a cycle.

Lemma 1. *Probe strongly chordal graphs are weakly chordal.*

Proof. Since probe strongly chordal graphs are probe chordal, they are perfect [15] and hence they have no induced odd hole [7]. A probe chordal graph also contains no antihole. Suppose there is an antihole. It has at most two nonprobes, and the edge joining them must be added because an antihole is not chordal.[3] However, the resulting graph still contains at least the complement of a path induced by five vertices, and hence a 4-cycle, and is still not chordal. Thus, it remains to be shown that there are no even holes.

Consider a graph G containing an induced even hole of length $2\mathsf{k}$, $\mathsf{k} \geq 3$. To make G chordal, one needs a set of k nonprobes. The even hole can be made into a sun by turning this set of nonprobes into a clique. However, if not all the edges between nonprobes are there, this even hole will be a trampoline. Since every trampoline contains a sun as an induced subgraph [6, 11], any embedding of this even hole will have a sun. Thus, G is not probe strongly chordal. □

Next we point out how to cope with chordless 4-cycles (the second step of our algorithm). To this end, we need some more notation and facts. A cycle in which probes and nonprobes alternate is called an *alternating cycle.*

Proposition 1 ([15]). *Let* $\mathsf{G} = (\mathbb{P} + \mathbb{N}, \mathsf{E})$ *be a partitioned graph. Let* C *be a chordless* 4*-cycle in* G*. If* G *is probe chordal, then* C *is alternating and any chordal embedding of* G *must have the edge filled in between the two nonprobes of* C*.*

Enhanced graphs [15] play a central role for the recognition of probe chordal graphs [3, 15] and are also a key concept in our recognition algorithm.

Definition 3 ([15]). *The* enhanced graph G^* *is obtained from a partitioned graph* $\mathsf{G} = (\mathbb{P}+\mathbb{N}, \mathsf{E})$ *by adding all edges between nonprobes in alternating chordless* 4*-cycles of* G*.*

As stated in the following theorem [15], the enhanced graph is the desired chordal embedding for the second step of the algorithm.

Theorem 3 ([15]). *Let* $\mathsf{G} = (\mathbb{P} + \mathbb{N}, \mathsf{E})$ *be a* PP*-chordal graph which is weakly chordal. Then the enhanced graph* G^* *is chordal.*

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[3] An antihole is not chordal for the following reason. A length-5 antihole is the same as a length-5 hole. Every antihole with at least 6 vertices contains 4 vertices which induce a chordless cycle.

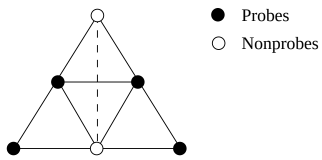

Fig. 2. A 3-sun in which one independent set vertex and one clique vertex are the only (nonadjacent) nonprobes. Adding the edge destroys the sun.

For the third and last step of our algorithm, by Theorem 2, it remains to destroy the induced suns in the enhanced graphs. As an example of destroying suns, take the 3-sun in which one independent set vertex and one clique vertex are the only (nonadjacent) nonprobes (see Fig. 2). The enhanced graph is the 3-sun itself, which is not strongly chordal. A strongly chordal embedding is obtained by adding the edge between the two nonprobes. The final and most important concept for our strategy to destroy suns is the notion of a "probe simple vertex."

Definition 4. *Let* $\mathsf{G} = (\mathbb{P} + \mathbb{N}, \mathsf{E})$ *be a partitioned graph. A vertex is* probe simple *if it can be made simple by adding some edges between nonprobes in* G*.*

Obviously, if G is PP-strongly chordal then, by Theorem 1, every induced subgraph has a probe simple vertex.

We summarize our findings in the following pseudo-code for recognizing PP-strongly chordal graphs. The correctness of the third and main step is heavily based on Theorem 4 shown subsequently.

Algorithm for recognizing PP-strongly chordal graphs
Input: Partitioned probe graph $\mathsf{G} = (\mathbb{P} + \mathbb{N}, \mathsf{E})$.
Output: Yes if G is PP-strongly chordal; otherwise, no.
1 **If** G is not weakly chordal **then** return no;
2 **If** there is a chordless 4-cycle that is not alternating **then** return no;
3 Construct the enhanced graph G^*;
While G^* is not empty **do**
  If G^* has a probe simple vertex s **then**
    Insert an inclusion-minimal set of edges into G^* to make s simple;
    $\mathsf{G}^* := \mathsf{G}^* - \mathsf{s}$
  else return no;
Return yes

In the following, given a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, a vertex $\nu \in \mathsf{V}$, and a graph $\mathsf{H} = (\mathsf{V_H}, \mathsf{E_H})$ with $\mathsf{V_H} \subseteq \mathsf{V} \setminus \{\nu\}$, we use $\mathsf{H} + \nu$ to denote the graph obtained by adding ν to H and adding the edges between ν and $\mathsf{N_G}(\nu) \cap \mathsf{V_H}$.

Theorem 4. *Let* s *be a probe simple vertex in an enhanced graph* G*. Let* G' *be the graph obtained from* G *by adding an inclusion-minimal set of edges between nonprobes to* G *in order to make* s *simple. Then,* G *is* PP*-strongly chordal if and only if* $\mathsf{G}' - \mathsf{s}$ *is* PP*-strongly chordal.*

Proof. (Sketch) "$\Leftarrow$": We show this direction by constructing from a strongly chordal embedding H of $G' - s$ a strongly chordal embedding of G. Since s is simple in G', $H + s$ is chordal. If $H + s$ is strongly chordal, then we are done. Hence, we consider the case that $H + s$ contains a sun: We transform H into a new, strongly chordal embedding $\hat{H}$ of $G - s$ such that s is simple in $\hat{H} + s$; that is, for every two vertices $x, y \in N_G(s)$ either $N_{\hat{H}}[x] \subseteq N_{\hat{H}}[y]$ or $N_{\hat{H}}[y] \subseteq N_{\hat{H}}[x]$. Then $\hat{H} + s$ is a strongly chordal embedding of G.

To construct $\hat{H}$, we apply the following operation to all pairs $x, y \in N_G(s)$ with $N_{G'}[x] \subseteq N_{G'}[y]$ while $N_H[x] - N_H[y] \neq \varnothing$. Obviously, $N_H[x] - N_H[y] \subseteq \mathbb{N}$. Consider removing all edges $\{x, z\}$ where $z \in N_H(x) - N_H[y]$. Assume this creates a chordless cycle C of length at least 4. Since H is chordal, $x \in C$. Then y is adjacent to the two neighbors of x in C and C contains a vertex z in $N_H(x) - N_H[y]$ which is not adjacent to y. It follows that the subgraph of H induced by $C - x + y$ contains a chordless cycle, which is a contradiction.

A similar argument shows that the resulting graph is *strongly* chordal. Repeated application of the above operation gives the graph $\hat{H}$.

"$\Rightarrow$": The basic idea of the proof of this direction is as follows. To show that $G' - s$ is PP-strongly chordal, we construct a strongly chordal embedding of $G' - s$ from the strongly chordal embedding H of G. More precisely, by adding and deleting edges, we will construct a graph $\hat{H}$ from H such that $\hat{H} - s$ is a strongly chordal embedding of $G' - s$. The central difficulty arising here is as follows. In order to make s simple in G', we have added edges to G to obtain G'. Among these edges there might be edges that are not present in H. Just adding these edges to H, however, does not necessarily guarantee that the resulting graph is strongly chordal. Thus, we will not only add these edges to H, but also delete and add some other edges when constructing $\hat{H}$ from H.

Without loss of generality let us assume the ordering $N_{G'}[s] \subseteq N_{G'}[x_1] \subseteq \cdots \subseteq N_{G'}[x_k]$ for $N_{G'}(s) = \{x_1, \ldots, x_k\}$. We write also $s = x_0$. The graph $\hat{H}$ shall have the following properties: $\hat{H}$ is strongly chordal (P1), $N_{\hat{H}}(s) = N_{G'}(s) = \{x_1, \ldots, x_k\}$ (P2), and $N_{\hat{H}}[s] \subseteq N_{\hat{H}}[x_1] \subseteq \cdots \subseteq N_{\hat{H}}[x_k]$ (P3).

Step 1. Apply the following operation for $i = 0, \ldots, k$: As long as x_i has a neighbor z in H for which there exists a probe vertex $y \in \{x_{i+1}, \ldots, x_k\} \cap \mathbb{P}$ with $z \notin N_H(y)$, remove the edge $\{x_i, z\}$ from H.

It can be shown that the resulting graph H_1 is strongly chordal.

Step 2. Turn $N_{H_1}[s]$ into a clique.

Let H_2 be the result. Assume H_2 has a chordless cycle C. Then C has exactly two vertices $x, y \in N_{H_1}(s)$. Then $C + s$ induces a chordless cycle in H_1 and this is a contradiction. Now let D be a component of $H_1 - N_{H_1}[s]$. Then any two vertices $x, y \in N_{H_1}(D)$ are adjacent and have no *private neighbors* in D. Herein, for two vertices x and y, a private neighbor of x is a vertex adjacent to x but not to y. It follows that a vertex in D which is simple in H_1 remains simple in H_2. Thus H_2 has a *simple elimination ordering*, *i.e.*, there is a vertex ordering $(v_1, \ldots, v_n)$ of H_2 such that, for all $1 \leq i \leq n$, the vertex v_i is simple in the subgraph of H_2 induced by $\{v_i, \ldots, v_n\}$. Thus, H_2 is strongly chordal.

Step 3. For $i = 0, \ldots, k-1$ and each $y \in \{x_{i+1}, \ldots, x_k\}$, make y adjacent to all vertices of $N_{H_2}[x_i]$.

Let H_3 be the resulting graph. Assume that making y adjacent to all vertices of $N_{H_2}[x_i]$ creates a chordless cycle C. Then for some $z \in N_{H_2}(x_i)$, $y, z \in C$. Now let z' be the other neighbor of y in C. Then $N_{H_2}(x_i) \cap C \subseteq \{y, z, z'\}$. It follows that $H_2[C + x_i]$ also contains a chordless cycle. Assume this step creates a "bad" cycle S, *i.e.*, an even cycle of length at least 6 without an odd chord. Let $y, z, z' \in S$ as above and let z'' be the other neighbor of z in S. Then x_i is adjacent to at least one of z', z'' and it follows that also H_2 has such a bad cycle, a contradiction.

Step 4. Remove the edges $\{s, z\}$ from H_3 for all $z \in N_{H_3}(s) - N_{G'}(s)$.

In the remaining graph $\hat{H}$, s has the same neighborhood as in G'. Notice that $\hat{H}$ is strongly chordal: Indeed, $H_3 - s = \hat{H} - s$, and so $\hat{H} - s$ is strongly chordal. The vertex s is simple in $\hat{H}$, thus also $\hat{H}$ is strongly chordal. □

The correctness of the algorithm as given in the pseudo-code above follows from Lemma 1, Proposition 1, Theorem 3, and Theorem 4. Altogether, we obtain the following main result.

Theorem 5. *It can be decided in polynomial time if a partitioned graph* G *is* PP*-strongly chordal. If so, also an embedding of* G *can be found.*

4 Partitioned Probe Chordal Bipartite Graphs

Recall that our main goal is to devise a polynomial-time algorithm for the recognition of probe totally balanced matrices. To this end, we make use of the fact that a $0/1$-matrix is totally balanced if and only if the corresponding bipartite graph is *chordal bipartite* [11].

In this section, we show that the recognition algorithm for PP-strongly chordal graphs can be used for recognizing PP-chordal bipartite graphs and indicate how this transfers to the recognition of probe totally balanced matrices.

One characterization of chordal bipartite graphs that is useful for our purposes is the following by Dahlhaus [9]. If $B = (X, Y, E)$ is a bipartite graph then we denote by $\mathsf{split}_X(B)$ the graph obtained from B by completing X into a clique.[4]

Theorem 6 ([9]). *A bipartite graph* $B = (X, Y, E)$ *is chordal bipartite if and only if* $\mathsf{split}_X(B)$ *is strongly chordal.*

From now on, let B be a partitioned probe bipartite graph. Without loss of generality, we assume that B is connected, since otherwise we may concentrate on the components individually. Obviously, we cannot simply apply the recognition algorithm for partitioned probe strongly chordal graphs to $\mathsf{split}_X(B)$, since the completion of X into a clique possibly adds edges between nonprobe vertices. Instead, we use the following trick: We add two probe vertices to Y, say α and ω,

[4] The operation $\mathsf{split}_X(B)$ transforms B into a *splitgraph*, that is, a graph which has a partition of its vertex set into a clique and an independent set.

and make these adjacent to all vertices of X. Let $\alpha\omega(B)$ be the resulting bipartite graph. Next, for every probe vertex x in X, we add edges between x and all other vertices in X. Since α and ω create a chordless 4-cycle with any pair of nonprobe vertices of X, any embedding of this new graph into a strongly chordal graph forces X into a clique. Let $\mathsf{PPsplit}_X(\alpha\omega(B))$ be the graph obtained from $\alpha\omega(B)$ by adding edges between vertices in X as described above. In the following, we present a "probe version" of Theorem 6. Afterwards, Theorem 8 justifies the application of the partitioned probe strongly chordal graph recognition algorithm from Section 3 to $\mathsf{PPsplit}_X(\alpha\omega(B))$.

Theorem 7. *A partitioned probe bipartite graph* $B = (X, Y, E)$ *is partitioned probe chordal bipartite if and only if* $\mathsf{PPsplit}_X(\alpha\omega(B)) = (X, Y \cup \{\alpha, \omega\}, E')$ *is partitioned probe strongly chordal.*

Combining Theorems 5 and 7, we arrive at the main result of this section.

Theorem 8. *Partitioned probe chordal bipartite graphs can be recognized in polynomial time.*

Proof. The algorithm works as follows: Given a partitioned probe bipartite graph $B = (X, Y, E)$, construct $\mathsf{PPsplit}_X(\alpha\omega(B)) = (X, Y \cup \{\alpha, \omega\}, E')$ as described above. The new graph is checked against being partitioned probe strongly chordal by the algorithm in Section 3. The correctness and the running time follow from Theorems 5 and 7. □

Corollary 1. *Probe totally balanced matrices can be recognized in polynomial time.*

5 Future Work

With this paper we try to initiate research on special matrix sandwich problems, that is, probe matrix problems. These stand in close relationship with partitioned probe graph problems. As an important starting case, we settled the complexity of the recognition problem of the probe totally balanced matrices, thereby also showing the polynomial-time recognizability of partitioned probe strongly chordal graphs and of partitioned probe chordal bipartite graphs. Note that the corresponding sandwich versions are NP-complete [12]. As to future work, we face the following two challenges (refer to [4, Chapter 9] for definitions).

1. Show that probe balanced matrices can be recognized in polynomial time. In their seminal work, Conforti, Cornuéjols, and Rao [8] designed a polynomial-time algorithm for recognizing balanced matrices (also see [17, 23]).
2. Show that probe totally unimodular matrices can be recognized in polynomial time. Here, Seymour's [22] famous decomposition result for totally unimodular matrices should be helpful (also see [21, Chapters 19–21]).

Further opportunities for future work include showing the polynomial-time recognizability of unpartitioned strongly chordal graphs and studying optimization versions of the probe problems considered. In the latter case, the natural task would be to minimize the number of edges added and the number of $\star$s turned into 1s, respectively.

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