

# The Parameterized Complexity of the Induced Matching Problem in Planar Graphs\*

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**Abstract.** Given a graph  $G$  and an integer  $k \geq 0$ , the NP-complete INDUCED MATCHING problem asks for an edge subset  $M$  such that  $M$  is a matching and no two edges of  $M$  are joined by an edge of  $G$ . The complexity of this problem on general graphs as well as on many restricted graph classes has been studied intensively. However, little is known about the parameterized complexity of this problem. Our main contribution is to show that INDUCED MATCHING, which is  $W[1]$ -hard in general, admits a linear problem kernel on planar graphs. Additionally, we generalize a known algorithm for INDUCED MATCHING on trees to graphs of bounded treewidth using an improved dynamic programming approach.

## 1 Introduction

A *matching* in a graph is a set of edges no two of which have a common endpoint. An *induced matching*  $M$  of a graph  $G = (V, E)$  is an edge-subset  $M \subseteq E$  such that (1)  $M$  is a matching and (2) no two edges of  $M$  are joined by an edge of  $G$ . In other words, the subgraph induced by  $V(M)$  is precisely the graph consisting of the edges in  $M$ . Let  $\text{im}(G)$  denote the size of a largest induced matching in  $G$ . The decision version of the INDUCED MATCHING problem asks, given a graph  $G$  and an integer  $k$ , whether  $G$  has an induced matching of size at least  $k$ . The optimization version asks for an induced matching of maximum size. The INDUCED MATCHING problem was introduced as a variant of the maximum matching problem and motivated by Stockmeyer and Vazirani [22] as the “risk-free” marriage problem<sup>3</sup>. This problem has been intensively studied in recent years. It is known to be NP-complete for planar graphs of maximum degree 4 [17], bipartite graphs of maximum degree 3,  $r$ -regular graphs for  $r \geq 5$ , line-graphs and

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<sup>3</sup> Find the maximum number of pairs such that each married person is compatible with no married person except the one he or she is married to.

Hamiltonian graphs [18]. The problem is polynomial time solvable for trees [23] and weakly chordal graphs [6]. There exist many other results on special graph classes (see, e.g., [5,12,19]). Regarding polynomial-time approximability, it is known that the INDUCED MATCHING problem is APX-hard on  $4r$ -regular graphs, for all  $r \geq 1$  [23], and bipartite graphs with maximum degree 3 [10]. Moreover, for  $r$ -regular graphs it is NP-hard to approximate INDUCED MATCHING within a factor of  $r/2^{O(\sqrt{\ln r})}$  [8]. There exists an approximation algorithm for the problem on  $d$ -regular graphs ( $d \geq 3$ ) with performance ratio  $d - 1$  [10], which has subsequently been improved to  $0.75d + 0.15$  [13].

In contrast to these results, little is known about the parameterized complexity of INDUCED MATCHING. To the best of our knowledge, the only known result is that the problem is  $W[1]$ -hard (with respect to the matching size as parameter) in the general case [20], and hence unlikely to be fixed-parameter tractable. Therefore, it is of interest to study the parameterized complexity of the problem in those restricted graph classes where it remains NP-complete. In this paper, we focus on planar graphs. The parameterized complexity of various NP-complete problems in planar graphs has already been studied. An interesting aspect of such studies are *linear* problem kernels. The intuitive idea behind kernelization is that a polynomial-time preprocessing step removes the “easy” parts of a problem instance such that only the “hard” core of the problem remains, which can then be solved by other methods. We call such a core a linear kernel if we can prove that its size is a linear function of the parameter. For a recent survey about problem kernelization, see [14].

Using a newly introduced technique, the question of whether DOMINATING SET has a linear kernel in planar graphs was answered positively by Alber et al. [2]. The kernel size has subsequently been improved by Chen et al. [7]. Moreover, they show lower bounds on the kernel size for DOMINATING SET, VERTEX COVER, and INDEPENDENT SET in planar graphs. The technique developed by Alber et al. [2] has been exploited by Guo et al. [16] in developing a linear kernel for FULL-DEGREE SPANNING TREE, a maximization problem. Moreover, Fomin and Thilikos [11] extended the technique to graphs of bounded genus. Very recently, Guo and Niedermeier [15] gave a generic kernelization framework for NP-hard problems in planar graphs based on that technique. Thus far, the technique has been applied to problems whose solutions are vertex subsets. We give the first application of this technique for a maximization problem whose solutions are edge subsets.

As our main result, we show that INDUCED MATCHING in planar graphs admits a linear problem kernel. We adapt and extend the known kernelization technique [2,16,15]. The corresponding data reduction rules can be carried out in linear time. Moreover, we generalize an algorithm for INDUCED MATCHING on trees by Zito [23] to graphs of bounded treewidth using an improved dynamic programming approach, which runs in  $O(4^\omega \cdot n)$  time, where  $\omega$  is the width of the given tree decomposition.

In Section 2, we start out with some basic definitions and notation. In Section 3, we present the kernelization proof, which is the main technical contribu-

tion of this paper. Finally, in Section 4, we outline the algorithm on graphs of bounded treewidth and give an outlook on possible future research.

## 2 Preliminaries

In this paper, we deal with fixed-parameter algorithms that emerge from the field of parameterized complexity [9,21]. A parameterized problem is *fixed-parameter tractable* if it can be solved in  $f(k) \cdot n^{O(1)}$  time, where  $f$  is a computable function depending only on the parameter  $k$ . A common method to prove that a problem is fixed-parameter tractable is to provide data reduction rules that lead to a problem kernel. Given a problem instance  $(I, k)$ , a *data reduction rule* replaces that instance by another instance  $(I', k')$  in polynomial time, such that  $(I, k)$  is a yes-instance iff  $(I', k')$  is a yes-instance. An instance to which none of a given set of data reduction rules applies is called *reduced* with respect to this set of rules. A parameterized problem is said to have a *problem kernel* if the resulting reduced instance has size  $f(k)$  for a function  $f$  depending only on  $k$ . If  $f(k) = c \cdot k$  for some constant  $c$ , then we call the kernel *linear*. The basic complexity class for fixed-parameter intractability is  $W[1]$  [9].

In this paper we assume that all graphs are simple and undirected. For a graph  $G$ , let  $V(G)$  denote its vertex set and  $E(G)$  denote its edge set. For a subset  $V' \subseteq V$ , let  $G[V']$  denote the subgraph of  $G$  induced by  $V'$ . Let  $G \setminus V' := G[V \setminus V']$ , and for  $v \in V$ , let  $G - v := G \setminus \{v\}$ . Let  $N(v) := \{u \in V : \{u, v\} \in E\}$  be the (*open*) *neighborhood* of  $v$ . We assume that paths are *simple*, that is, each vertex appears at most once in a path. A path  $P$  from  $a$  to  $b$  is denoted as a vector  $P = (a, \dots, b)$ , and  $a$  and  $b$  are called the *endpoints* of  $P$ . The *length* of a path  $(a_1, a_2, \dots, a_q)$  is  $q-1$ , that is, the number of edges on it. For an edge set  $M$  we define  $V(M) := \bigcup_{e \in M} e$ . The *distance*  $d(u, v)$  between two vertices  $u, v$  is the length of a shortest path between them. The *distance* between two edges  $e_1, e_2$  is the minimum distance between two vertices  $v_1 \in e_1$  and  $v_2 \in e_2$ .

If a graph can be drawn in the plane without edge crossings then it is *planar*. A *plane graph* is a planar graph with a fixed embedding in the plane. Given a plane graph, a cycle  $C = (a, \dots, a)$  of length at least three encloses an *area*  $A$  of the plane. The cycle  $C$  is called the *boundary* of  $A$ , all vertices in the area  $A$  are *inside*  $A$ . A vertex is *strictly inside*  $A$  if it is inside  $A$  and not part of  $C$ .

## 3 A Linear Kernel on Planar Graphs

In order to show our kernel, we employ the following data reduction rules. Compared to the data reduction rules applied in other proofs of planar kernels [2,7,16], these data reduction rules are quite simple and can be carried out in  $O(n + m)$  time on general graphs and thus in  $O(n)$  time in planar graphs.

- (R0) *Degree Zero Rule*: Delete vertices of degree zero.
- (R1) *Degree One Rule*: If a vertex  $u$  has two distinct neighbors  $x, y$  of degree 1, then delete  $x$ .

**(R2) Degree Two Rule:** If  $u$  and  $v$  are two vertices such that  $|N(u) \cap N(v)| \geq 2$  and if there exist two vertices  $x, y \in N(u) \cap N(v)$  with  $\deg(x) = \deg(y) = 2$ , then delete  $x$ .

Note that these data reduction rules are parameter-independent. The following is our main theorem whose proof spans the remainder of this section.

**Theorem 1.** *Let  $G = (V, E)$  be a planar graph reduced with respect to the rules R0, R1, and R2. Then  $|V| \leq c \cdot \text{im}(G)$  for some constant  $c$ . That is, the MAXIMUM INDUCED MATCHING problem in planar graphs admits a linear problem kernel.*

The basic observation is that if  $M$  is a maximum induced matching of a graph  $G = (V, E)$  then for each vertex  $v \in V$  there exists a  $u \in V(M)$  such that  $d(u, v) \leq 2$ . Otherwise, we could add edges to  $M$  and obtain a larger induced matching. Since every vertex in the graph is within distance at most two to some vertex in  $V(M)$ , we know, roughly speaking, that the edges in  $M$  have distance at most four to other edges in  $M$ . This leads to the idea of regions “in between” matching edges that are close to each other. We will see that these regions cannot be too large if the graph is reduced with respect to the above data reduction rules. Moreover, we show that there cannot be many vertices that are not contained within such regions.

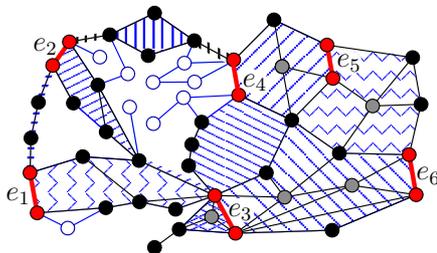
This idea of a region decomposition was introduced in [2], but the definition of a region as it appears there is much simpler since the regions are defined between vertices, and they are smaller. The remaining part of this section is dedicated to the proof of Theorem 1. First, in Section 3.1 we show how to find a “maximal region decomposition” of a reduced graph that contains only  $O(|M|)$  regions, where  $M$  is the size of a maximum induced matching of the graph. Then, in Section 3.2 we show that a region in such a maximal region decomposition contains only a constant number of vertices. Finally, in Section 3.3 we show that in any reduced graph there are only  $O(|M|)$  vertices which lie outside of regions.

### 3.1 Finding a Maximal Region Decomposition

**Definition 1.** *Let  $G$  be a plane graph and  $M$  a maximum induced matching of  $G$ . For edges  $e_1, e_2 \in M$ , a region  $R(e_1, e_2)$  is a closed subset of the plane such that*

1. *the boundary of  $R(e_1, e_2)$  is formed by two length-at-most-four paths*
  - $(a_1, \dots, a_2)$ ,  $a_1 \neq a_2$ , between  $a_1 \in e_1$  and  $a_2 \in e_2$ ,
  - $(b_1, \dots, b_2)$ ,  $b_1 \neq b_2$ , between  $b_1 \in e_1$  and  $b_2 \in e_2$ , and*by  $e_1$  if  $a_1 \neq b_1$  and  $e_2$  if  $a_2 \neq b_2$ ;*
2. *for each vertex  $x$  in the region  $R(e_1, e_2)$ , there exists a  $y \in V(\{e_1, e_2\})$  such that  $d(x, y) \leq 2$ ;*
3. *no vertices inside the region other than endpoints of  $e_1$  and  $e_2$  are from  $M$ .*

*The set of boundary vertices of  $R$  is denoted by  $\delta R$ . We write  $V(R(e_1, e_2))$  to denote the set of vertices of a region  $R(e_1, e_2)$ , that is, all vertices strictly inside  $R(e_1, e_2)$  together with the boundary vertices  $\delta R$ . A vertex in  $V(R(e_1, e_2))$  is inside  $R$ .*



**Fig. 1.** An example of an  $M$ -region decomposition: black vertices denote boundary vertices; gray vertices lie strictly inside a region and white vertices lie outside of regions. Each region is hatched with a different pattern. Note the special cases, as for instance regions that consist of a path like the region between  $e_1$  and  $e_2$ , or regions that are created by only one matching edge (the region on the left side of  $e_3$ ).

Note that the two enclosing paths may be identical; the corresponding region then consists solely of a simple path of length at most four. Note also that  $e_1$  and  $e_2$  may be identical.

**Definition 2.** An  $M$ -region decomposition of  $G = (V, E)$  is a set  $\mathcal{R}$  of regions such that no vertex in  $V$  lies strictly inside more than one region from  $\mathcal{R}$ . For an  $M$ -region decomposition  $\mathcal{R}$ , we define  $V(\mathcal{R}) := \bigcup_{R \in \mathcal{R}} V(R)$ . An  $M$ -region decomposition  $\mathcal{R}$  is maximal if there is no  $R \notin \mathcal{R}$  such that  $\mathcal{R} \cup \{R\}$  is an  $M$ -region decomposition with  $V(\mathcal{R}) \subsetneq V(\mathcal{R}) \cup V(R)$ .

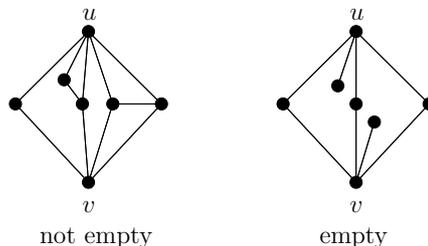
For an example of an  $M$ -region decomposition, see Fig. 1.

**Lemma 1.** Given a plane reduced graph  $G = (V, E)$  and a maximum induced matching  $M$  of  $G$ , there exists an algorithm that constructs a maximal  $M$ -region decomposition with  $O(|M|)$  regions.

Lemma 1 can be proved by exhibiting a greedy algorithm that builds a maximal  $M$ -region decomposition in a stepwise manner by searching a region of maximal size that is not yet in the region decomposition at the actual step of the algorithm. Since this approach is similar to the algorithms by Alber et al. [2] and Guo et al. [16], we omit the details here.

### 3.2 Bounding the Size of a Region

To upper-bound the size of a region  $R$  we make use of the fact that any vertex strictly inside  $R$  has distance at most two from some vertex in  $\delta R$ . For this reason, the vertices strictly inside  $R$  can be arranged in two layers. The first layer consists of the neighbors of boundary vertices, and the second of all the remaining vertices, that is, all vertices at distance at least two from every boundary vertex. The proof strategy is to show that if any of these layers contains too



**Fig. 2.** A diamond (left) and an empty diamond (right) in a reduced plane graph.

many vertices then there exists an induced matching  $M'$  with  $|M'| > |M|$ . An important structure for our proof are areas enclosed by 4-cycles, called *diamonds*.

**Definition 3.** Let  $u$  and  $v$  be two vertices in a plane graph. A diamond<sup>4</sup> is a closed area of the plane with two length-2 paths between  $u$  and  $v$  as boundary. A diamond  $D(u, v)$  is empty, if every edge  $e$  in the diamond is incident to either  $u$  or  $v$ .

Fig. 2 shows an empty and a non-empty diamond. In a reduced plane graph empty diamonds have a restricted size. We are especially interested in the maximum number of vertices strictly inside an empty diamond  $D(u, v)$  that have both  $u$  and  $v$  as neighbors. The following lemma is easy to show.

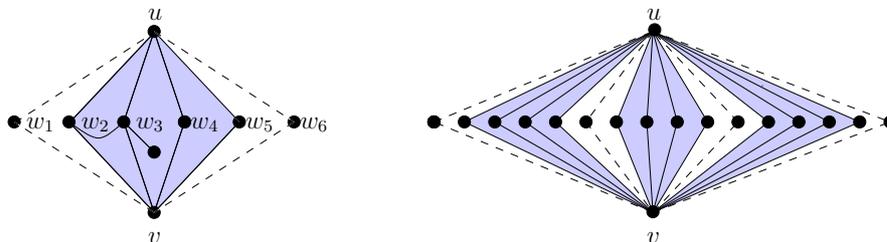
**Lemma 2.** Let  $D(u, v)$  be an empty diamond in a reduced plane graph. Then there exists at most one vertex strictly inside  $D(u, v)$  that has both  $u$  and  $v$  as neighbors.

Lemma 2 shows that if there are more than three edge-disjoint length-two paths between two vertices  $u, v$ , then there must be an edge  $e$  in an area enclosed by two of these paths such that  $e$  is not incident to  $u$  or  $v$ . This fact is used in the following lemma to show that the number of length-two paths between two vertices of a reduced plane graph is bounded.

**Lemma 3.** Let  $u$  and  $v$  be two vertices of a reduced plane graph  $G$  such that there exists two distinct length-2 paths  $(u, x, v)$  and  $(u, y, v)$  enclosing an area  $A$  of the plane. Let  $M$  be a maximum induced matching of  $G$ . If neither  $x$  nor  $y$  is an endpoint of an edge in  $M$  and no vertex strictly inside  $A$  is contained in  $V(M)$ , then the following holds:

If neither  $u$  nor  $v$  is an endpoint of an edge in  $M$ , then there are at most 5 edge-disjoint length-2 paths between  $u$  and  $v$  inside  $A$ . If exactly one of  $u$  or  $v$  is an endpoint of an edge in  $M$ , then there are at most 10 such paths, and if both  $u$  and  $v$  are endpoints of edges in  $M$ , then there are at most 15 such paths.

<sup>4</sup> In standard graph theory, a diamond denotes a 4-cycle with exactly one chord. We abuse this term here. Note that diamonds also play an important role in proving linear problem kernels in planar graphs for other problems [2,15].



**Fig. 3.** Left: An embedding of the vertices  $w_1, \dots, w_6$  for the first case in the proof of Lemma 3. Right: An embedding of 16 neighbors of  $u$  and  $v$  for the last case of the proof. The diamonds are shaded and the “isolation paths” are drawn with dashed lines.

*Proof.* The idea is to show that if there are more than the claimed number of length-2 paths between  $u$  and  $v$ , then we can exhibit an induced matching  $M'$  with  $|M'| > |M|$ , which would then contradict the optimality of  $M$ .

First, we consider the case when neither  $u$  nor  $v$  is contained in  $V(M)$ . Suppose for the purpose of contradiction that there are 6 common neighbors  $w_1, \dots, w_6$  of  $u$  and  $v$  that lie inside  $A$  (that is, strictly inside and on the enclosing paths). Without loss of generality, suppose that these vertices are embedded as shown in Fig. 3 (left-hand side), with  $w_1$  and  $w_6$  lying on the enclosing paths. Consider the diamond  $D$  with the boundary induced by the vertices  $u, v, w_2, w_5$ . Since  $w_3$  and  $w_4$  are strictly inside  $D$  and are incident to both  $u$  and  $v$ , by Lemma 2, we know that  $D$  is not empty. That is, there exists an edge  $e$  in  $D$  which is not incident to  $u$  or  $v$ . Clearly  $e$  is incident to neither  $w_1$  nor  $w_6$  and the endpoints of  $e$  are at distance at least 2 from every vertex in  $V(M)$ . Therefore, we can add  $e$  to  $M$  and obtain a larger induced matching, which contradicts the optimality of  $M$ .

Next, consider the case when exactly one of  $u$  or  $v$  is an endpoint of an edge  $e$  in  $M$ . Using the same idea as above, it is easy to see that if there exist 11 length-2 paths between  $u$  and  $v$ , then there are at least two non-empty diamonds (using  $(u, w_1, v)$ ,  $(u, w_6, v)$  and  $(u, w_{11}, v)$  as “isolation paths”) whose boundaries share only  $u$  and  $v$ . We can then replace  $e$  in  $M$  by edges  $e_1$  and  $e_2$ , one from each nonempty diamond, and obtain a larger induced matching.

The last case, when both  $u$  and  $v$  are endpoints of edges in  $M$ , can be handled in the same way using three non-empty diamonds (see Fig. 3). □

Lemma 3 is needed to upper-bound the number of vertices inside and outside of regions that are connected to at least two boundary vertices.

The next two lemmas are needed to upper-bound the number of vertices that are connected to exactly one boundary vertex. First, Lemma 4 upper-bounds the number of such vertices under the condition that they are contained in an area which is enclosed by a short cycle. Lemma 4 is then used in Lemma 5 to upper-

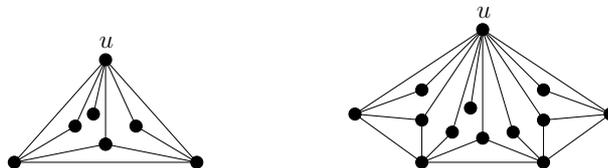


Fig. 4. Worst-case embeddings to illustrate Lemma 4.

bound the total number of such vertices for a given boundary vertex. The proofs of both lemmas are omitted due to space restrictions.

**Lemma 4.** *Let  $u$  be a vertex in a reduced plane graph  $G$  and let  $v, w \in N(u)$  be two distinct vertices that have distance at most three in  $G - u$ . Let  $P$  denote a shortest path between  $v$  and  $w$  in  $G - u$  and let  $A$  denote the area of the plane enclosed by  $P$  and the path  $(v, u, w)$ . If there are at least 9 neighbors of  $u$  strictly inside  $A$ , then there is at least one edge strictly inside  $A$ .*

Fig. 4 shows, for two different situations, the maximum number of neighbors of  $u$  that can be strictly inside  $A$  such that no edge lies strictly inside  $A$ .

**Lemma 5.** *Let  $u$  be a boundary vertex of a region  $R(e_1, e_2)$  in a reduced plane graph  $G$ , and let  $M$  be a maximum induced matching of  $G$ . If  $u$  has at least 41 neighbors strictly inside  $R$  that are not adjacent to any other boundary vertex, then we can find an induced matching  $M'$  with  $|M'| > |M|$ .*

Using Lemma 3 and Lemma 5, we can now upper-bound the number of vertices inside a region.

**Lemma 6.** *A region  $R(e_1, e_2)$  of an  $M$ -region decomposition of a reduced plane graph contains  $O(1)$  vertices.*

*Proof.* We prove the lemma by partitioning the vertices strictly inside  $R(e_1, e_2)$  into two sets  $A$  and  $B$ , where  $A$  consists of all vertices at distance exactly one from some boundary vertex, and  $B$  consists of all vertices at distance at least two from every boundary vertex, and then showing that  $|A|$  and  $|B|$  are upper-bounded by a constant.

To this end, partition  $A$  into  $A_1$  and  $A_2$ , where  $A_1$  contains all vertices in  $A$  that have exactly one neighbor on the boundary, and  $A_2$  all vertices that have at least two neighbors on the boundary. To upper-bound the size of  $A_1$ , observe that due to Lemma 5, a vertex  $v \in \delta R$  on the boundary can have at most 41 neighbors in  $A_1$ . Since a region has at most ten boundary vertices, we conclude that  $A_1$  contains at most 410 vertices.

Next we upper-bound the size of  $A_2$ . Consider the planar graph  $G'$  induced by  $\delta R \cup A_2$ . Every vertex in  $A_2$  is adjacent to at least two boundary vertices in  $G'$ . Replace every vertex  $v \in A_2$  with an edge connecting two arbitrary neighbors of  $v$  on the boundary. Merge multiple edges between two boundary vertices into a single edge. Since  $G'$  is planar, the resulting graph must also be planar.

As  $|\delta R| \leq 10$ , using the Euler formula we conclude that the resulting graph has at most  $3 \cdot 10 - 6 = 24$  newly added edges. By Lemma 3, each such edge represents at most 15 length-two paths, and thus  $|A_2| \leq 24 \cdot 15 = 360$ .

To upper-bound the size of  $B$ , observe that  $G[B]$  must be a graph without edges (that is,  $B$  is an independent set). By the Degree One Rule, each vertex in  $A$  has at most one neighbor in  $B$  of degree one. Therefore, there are  $O(1)$  degree-one vertices in  $B$ . To bound the number of degree-at-least-two vertices in  $B$ , we use the same argument as the one used to bound the size of  $A_2$ . Since  $|A| = O(1)$ , there is a constant number of degree-at-least-two vertices in  $B$ . Therefore  $|B| = O(1)$ . This completes the proof.  $\square$

**Proposition 1.** *Let  $G$  be a reduced plane graph and let  $M$  be a maximum induced matching of  $G$ . There exists an  $M$ -region decomposition such that the total number of vertices inside all regions is  $O(|M|)$ .*

*Proof.* Using Lemma 1, there exists a maximal  $M$ -region decomposition for  $G$  with at most  $O(|M|)$  regions. By Lemma 6, each region has a constant number of vertices. Thus there are  $O(M)$  vertices inside regions.  $\square$

### 3.3 Bounding the Number of Vertices Lying Outside of Regions

In this section, we upper-bound the number of vertices that lie outside of regions of a maximal  $M$ -region decomposition. The strategy to prove this bound is similar to that used in the last section. We subdivide the vertices lying outside of regions into several disjoint subsets and upper-bound their sizes separately.

Note again that the distance from any vertex of the graph to a vertex in  $V(M)$  is at most two. We partition the vertices lying outside of regions into two sets  $A$  and  $B$ , where  $A$  is the set of vertices at distance exactly one from some vertex in  $V(M)$ , and  $B$  is the set of vertices at distance at least two from every vertex in  $V(M)$ . We bound the sizes of these two sets separately.

Partition  $A$  into two subsets  $A_1$  and  $A_2$ , where  $A_1$  is the set of vertices that have exactly one boundary vertex as neighbor, and  $A_2$  is the set of vertices that have at least two boundary vertices as neighbors. Note that each vertex  $v$  in  $A$  can be adjacent to exactly one vertex  $u \in V(M)$ . For if it is adjacent to distinct vertices  $u, w \in V(M)$ , then the path  $(u, v, w)$  can be added to the region decomposition, contradicting its maximality (recall that regions can consist of simple paths between two vertices in  $V(M)$ ). To bound the number of vertices in  $A_1$  we need the following lemma, which is easy to prove.

**Lemma 7.** *Let  $v$  be a vertex in  $A_1$  and let  $u$  be its neighbor in  $V(M)$ . Then for all  $w \in V(M) \setminus \{u\}$ , the distance between  $v$  and  $w$  in  $G - u$  is at least three.*

**Lemma 8.** *Given a maximal  $M$ -region decomposition consisting of  $O(|M|)$  regions, the set  $A$  contains  $O(|M|)$  vertices.*

*Proof.* To bound the size of  $A_1$ , we claim that each vertex  $u \in V(M)$  has at most 20 neighbors in  $A_1$ . Suppose, for the purpose of contradiction, that 21

vertices  $v_1, \dots, v_{21}$  in  $A_1$  are adjacent to  $u \in V(M)$ . Also assume that they are embedded in a clockwise fashion around  $u$  in that order. Let  $e$  be the edge in  $M$  incident to  $u$ . First, suppose that  $v_1$  and  $v_{11}$  have distance at least four in  $G - u$ . Then there exist edges  $e_a, e_b$  in  $G - u$  incident to  $v_1$  and  $v_{11}$ , respectively, that form an induced matching of size 2. Moreover by Lemma 7, the endpoints of  $e_a$  and  $e_b$  are not adjacent to any vertex of  $V(M)$  in  $G - u$ . Therefore,  $M' = (M \setminus \{e\}) \cup \{e_a, e_b\}$  is an induced matching of size larger than that of  $M$ , a contradiction to the maximum cardinality of  $M$ . The same holds if the distance between  $v_{11}$  and  $v_{21}$  is at least four in  $G - u$ . Therefore assume that in the graph  $G - u$ ,  $d(v_1, v_{11}) \leq 3$  and  $d(v_{11}, v_{21}) \leq 3$ . Let  $P_1$  and  $P_2$  be shortest paths in  $G - u$  between  $v_1$  and  $v_{11}$  and between  $v_{11}$  and  $v_{21}$ , respectively. Note that due to Lemma 7 these two paths cannot contain any vertex from  $V(M)$ . By Lemma 4, the areas enclosed by  $P_1$  and  $(v_1, u, v_{11})$ , and  $P_2$  and  $v_{11}, u, v_{21}$ , respectively, contain an edge strictly inside them. The edge  $e$  can be replaced by these two edges to obtain an induced matching of size larger than  $M$ , a contradiction to the maximum cardinality of  $M$ . This proves our claim. Since there are exactly  $2|M|$  vertices in  $V(M)$ , this shows that the total number of vertices in  $A_1$  is at most  $40|M|$ .

Next, we bound the size of  $A_2$ . Every vertex  $v$  in  $A_2$  is adjacent to a vertex  $u \in V(M)$  and some boundary vertex  $w \notin V(M)$ . Vertex  $w$  must be adjacent to  $u$ , for otherwise there is a path consisting of the vertices  $(u, v, w)$  and some subpath on the boundary where  $w$  lies which can be added to the region decomposition  $\mathcal{R}$ , contradicting its maximality. Since there are  $O(|M|)$  regions, there are  $O(|M|)$  possible boundary vertices adjacent to a vertex in  $V(M)$ . By Lemma 3, given a vertex  $x \in V(M)$  and  $y \in V \setminus V(\mathcal{R})$  there can be at most 10 vertices adjacent to both  $x$  and  $y$ . This shows that  $A_2$  contains  $O(|M|)$  vertices.  $\square$

It remains to bound the number of vertices in  $B$ , that is, the number of vertices outside of regions that are at distance at least two from every vertex in  $V(M)$ .

**Lemma 9.** *Given a maximal  $M$ -region decomposition with  $O(|M|)$  regions, the set  $B$  contains  $O(|M|)$  vertices.*

*Proof.* To bound the size of  $B$ , observe that  $G[B]$  is a graph without edges. Furthermore, observe that  $N(B) \subseteq A \cup A'$ , where  $A'$  is the set of boundary vertices in the  $M$ -region decomposition that are different from  $V(M)$ . By Lemma 8 and since the boundary of each region contains a constant number of vertices, the set  $C := A \cup A'$  contains  $O(|M|)$  vertices.

First, consider the vertices in  $B$  that have degree one. Obviously, there can be at most  $|C|$  such vertices due to the Degree One Rule. The remaining vertices are adjacent to at least two vertices in  $C$ . We can use an argument similar to the one used in the proof of Lemma 6 (using the Euler formula) to show that there are  $O(|C|)$  degree-at-least-two vertices in  $B$ . Thus,  $|B| = O(|C|) = O(|M|)$ .  $\square$

**Proposition 2.** *Given a maximal  $M$ -region decomposition with  $O(|M|)$  regions, the number of vertices that lie outside of regions is  $O(|M|)$ .*

*Proof.* Follows from Lemmas 8 and 9.  $\square$

Using Propositions 1 and 2, we can show that, given a reduced plane graph  $G$  and a maximum induced matching  $M$  of  $G$ , there exists an  $M$ -region decomposition with  $O(|M|)$  regions such that the number of vertices inside and outside of regions is  $O(|M|)$ . This shows the  $O(|M|)$  upper bound on the number of vertices as claimed in Theorem 1.

## 4 Further Results and Outlook

Zito [23] developed a linear-time dynamic programming algorithm to solve INDUCED MATCHING on trees. We generalize this approach to obtain a linear-time algorithm on graphs of bounded treewidth [4]. It is relatively easy to see that a standard dynamic programming approach would result in a running time of  $O(9^\omega \cdot n)$ , where  $\omega$  is the width of the given tree decomposition. With an improved dynamic programming algorithm, we obtain a running time of  $O(4^\omega \cdot n)$ . The description of this algorithm, which is inspired by a similar result for DOMINATING SET [3], is omitted due to space restrictions.

**Theorem 2.** *Let  $G = (V, E)$  be a graph with a given nice tree decomposition  $(\{X_i \mid i \in I\}, T)$ . Then, the size of a maximum induced matching of  $G$  can be computed in  $O(4^\omega \cdot n)$  time, where  $n := |I|$  and  $\omega$  denotes the width of the tree decomposition.*

As our main result, we have shown that INDUCED MATCHING in planar graphs admits a linear problem kernel. The data reduction rules for the planar case are very simple and the kernelization can be done in linear time.

A possible future research topic could be search tree algorithms for planar graphs. For DOMINATING SET in planar graphs, there exists a search tree algorithm [1], and it is open whether a similar result for INDUCED MATCHING on planar graphs is possible. Investigating the parameterized complexity of INDUCED MATCHING on other restricted classes of graphs may be of interest. We can show simple problem kernelizations for bounded-degree graphs, graphs of girth at least 6,  $C_4$ -free bipartite graphs, and line graphs. A class of major interest are bipartite graphs, where the parameterized complexity of INDUCED MATCHING is open.

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